Dyson Brownian Motion as a Limit of the Whittaker 2d Growth Model

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***Abstract*—This paper proves that a class of scaled Whittaker growth models will converge in distribution to the Dyson Brownian motion. A Whittaker 2d growth model is a continuous-time Markov diffusion process embedded on a spatial triangular array. Our result is interesting because each particle in a Whittaker 2d growth model only interacts with its neighboring particles, while each particle in the Dyson Brownian motion interacts with all the other particles. We provide two different proofs of the main result.**

***Index Terms*—Convergence, Dyson Brownian motion, Markov diffusion process, Whittaker 2d growth model.**

I. Introduction

A Whittaker 2d growth model is a continuous-time Markov diffusion process

$$T(t) = \{T_{k,j}(t),\, t > 0\}_{1\leq j\leq k\leq N} \in \mathbb{R}^{N(N+1)/2},$$

defined by the stochastic differential equations:

$$\begin{aligned}
&dT_{1,1} = dW_{1,1} + a_1 dt,\\
&\text{For } k = 2, \dots, N:\\
&dT_{k,1} = dW_{k,1} + (a_k + e^{T_{k-1,1}-T_{k,1}})dt,\\
&dT_{k,2} = dW_{k,2} + (a_k + e^{T_{k-1,2}-T_{k,2}} - e^{T_{k,2}-T_{k-1,1}})dt,\\
&\qquad \cdots\\
dT_{k,k-1} &= dW_{k,k-1} + (a_k + e^{T_{k-1,k-1}-T_{k,k-1}} - e^{T_{k,k-1}}\\
&\qquad - T_{k-1,k-2})dt,\\
&dT_{k,k} = dW_{k,k} + (a_k - e^{T_{k,k}-T_{k-1,k-1}})dt,
\end{aligned}$$

where $\{W_{k,i} : 1 \leq i \leq k \leq N\}$ are independent Brownian motions, and $\{a_k : 1 \leq k \leq N\}$ are constants. A Whittaker 2d growth model can be seen as a triangular array of spatially interacting particles.

From the above definition, a Whittaker 2d growth model consists of the Brownian motion diffusion and drift terms that characterize the interactions among $T_{k,j}$'s. A unique aspect of the model is that each drift term has at most two particles on the layer above. Also, the interactions are one-directional, meaning that each level can influence the next level but not the other way around.

The Whittaker 2d growth models are closely connected with several other fundamental mathematical and physical objects, such as the quantum toda lattice [1], the Whittaker functions [2], the q-Whittaker 2d growth model, and the Macdonald symmetric functions [3].

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The Dyson Brownian motion, introduced by in [4], is an important stochastic process. It is deeply connected with the random matrix theory (see, e.g., [5]–[11]). The existence and uniqueness of Dyson Brownian motion were summarized in [12]. An interlaced version of Dyson Brownian motion was developed in [13]. An infinite-dimensional Dyson Brownian motion was studied in [14]. A system of stochastic processes by generalizing the drift terms in the Dyson Brownian motion was shown to converge to the Wigner Law [15]. The Dyson Brownian motion was also shown to be closely connected with several other stochastic processes, including the non-colliding Brownian motion and the stochastic process of reflections [16].

In this paper, we will show that a scaled Whittaker 2d growth model, where $T_{k,j}(t)$ is replaced with $\frac{1}{\sqrt{\gamma}}T_{k,j}(\gamma t)$, converges in distribution to the Dyson Brownian motion as $\gamma \to \infty$. The result is interesting and somewhat surprising because each particle in a Whittaker 2d growth model only interacts with its neighboring particles; in contrast, each particle in the Dyson Brownian motion simultaneously interacts with all the other particles.

A different scaling of the Whittaker 2d growth model was studied in [17], where $T_{k,j}(t)$ was replaced with $\frac{1}{\gamma}T_{k,j}(\gamma t)$, and a large deviation principle was derived. It was shown that under that scaling, the Whittaker 2d growth model converges to the constant zero.

We will provide two different proofs of the main result. The first proof is from the perspective of the infinitesimal generator of Whittaker 2d growth models. The proof is based on the critical observation that under the above scale (of γ), the scaling effect will not essentially change the diffusion terms, but it will drive the drift terms to converge to the drift terms in the Dyson Brownian motion. Accordingly, the proof of convergence is based on elementary but highly nontrivial derivations of the scaled drift terms. The second proof is from the perspective of transition kernels. We will treat both the Whittaker 2d growth model and the Dyson Brownian motion as a consequence of their respective transition kernels and turn the problem into the proof of transition kernels.

The outline of the paper is given below. In Section II, we introduce the main result. In Section III, we provide proof of the main result. In section IV, we prove the main theorem in a different way by understanding the distribution through transition kernels.

II. Main Result

A. Scaled Whittaker 2d growth model

We show that a class of scaled Whittaker 2d growth models converge (in distribution) to the Dyson Brownian motion. In particular, we consider the Whittaker 2d growth models where $a_k = 0$ for all k, and $T_{k,j}(t)$ is replaced with $\frac{1}{\sqrt{\gamma}} T_{k,j}(\gamma t)$ for a positive scaling factor γ. Consequently, we have the scaled stochastic differential equation system

$$dT_{1,1} = dW_{1,1}$$

For $k = 2, 3. \dots, N$,

$$\begin{aligned}
dT_{k,1} &= dW_{k,1} + e^{\sqrt{\gamma}(T_{k-1,1} - T_{k,1})} dt, \\
dT_{k,2} &= dW_{k,2} + (e^{\sqrt{\gamma}(T_{k-1,2} - T_{k,2})} - e^{\sqrt{\gamma}(T_{k.2} - T_{k-1,1})}) dt, \\
&\dots \\
dT_{k,k-1} &= dW_{k,k-1} + (e^{\sqrt{\gamma}(T_{k-1,k-1} - T_{k,k-1})} \\
&\quad - e^{\sqrt{\gamma}(T_{k.k-1} - T_{k-1,k-2})}) dt, \\
dT_{k,k} &= dW_{k,k} - e^{\sqrt{\gamma}(T_{k,k} - T_{k-1,k-1})} dt.
\end{aligned}$$

where $\{W_{k,j}\}_{1 \le j \le k \le N}$ is a set of independent standard Brownian motions. We also suppose that the starting positions satisfy

$$T_{k+1,j+1}(0) < T_{k,j}(0) < T_{k+1,j}(0), 1 \le j \le k \le N. \quad (1)$$

The above constraint ensures that the process does not diverge.

B. Dyson Brownian motion

The Dyson Brownian motion is defined as follows. Let $(W_1, ..., W_N)$ be an N-dimensional Brownian motion in a probability space (Ω, P) equipped with a filtration $F = \{F_t, t \ge 0\}$. Let Δ_N be the set

$$\Delta_N = \{(x_i)_{1 \le i \le N} \in \mathbb{R}^N : x_1 < x_2 < \dots < x_{N-1} < x_N\}.$$

Let $\beta \ge 1$ be a constant and $t \ge 0$. Suppose that the initial condition is $\lambda_N(0) = (\lambda_N^1(0), \lambda_N^2(0), \cdots, \lambda_N^N(0)) \in \Delta_N$. It can be shown that there exists a unique strong solution to the stochastic differential system

$$d\lambda_N^i = \frac{\sqrt{2}}{\sqrt{\beta N}} dW^i + \frac{1}{N} \sum_{1 \le j \le N, j \ne i} \frac{1}{\lambda_N^i - \lambda_N^j} dt, \quad (2)$$

and that $\lambda_N(t) \in \Delta_N$ for all $t > 0$ (see, e.g., [12]). The process defined in (2) is named the Dyson Brownian motion.

With a different scaling and change of notation, Dyson Brownian motion can be rewritten as

$$dT_N^i = dW^i + \sum_{1 \le j \le N, j \ne i} \frac{1}{T_N^i - T_N^j} dt. \quad (3)$$

C. Main result: convergence of scaled Whittaker 2d growth models to the Dyson Brownian motion

Theorem II.1 (Main Theorem)**.** *Assume that the initial condition (1) hold. The Whittaker 2d growth models (in II-A) converge in distribution to the Dyson Brownian motion (3) as $\gamma \to \infty$.*

III. Proof of Theorem II.1

We provide a sketch of the proof. First, we will use O'Connel's approach in [1] to obtain an explicit form of the Whittaker 2d growth model's drift terms. We will then prove that the scaled drift terms will converge to the drift terms in the Dyson Brownian motion. In that proof, we will write the drift term as a fraction and separately derive the limit of its numerator and denominator.

We define $\Psi_0 : \mathbb{R}^{N+1} \to \mathbb{R}$ by

$$\Psi_0(x) = \int_{\mathbb{R}^{N(N+1)/2}} e^{F_0(T)} \prod_{k=1}^{N} \prod_{i=1}^{k} dT_{k,i}, \quad (4)$$

where $T_{N+1,i} = x_i$ for $1 \le i \le N+1$, and

$$F_0(T) = - \sum_{1 \le i \le k \le N} \left(e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}} \right).$$

In the sequel, we will also write

$$\prod_{k=1}^{N} \prod_{i=1}^{k} dT_{k,i}$$

as dT. To simplify the notation, we also define

$$G_k(T) = \sum_{i=1}^{k} e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}},$$

which implies that

$$F_0(T) = - \sum_{k=1}^{N} G_k(T).$$

It was shown in [1] that a Whittaker 2d growth model has the infinitesiaml generator

$$\frac{1}{2} \Delta + \nabla \log \Psi_0 \cdot \nabla, \quad (5)$$

where $\Delta = \sum_i \partial_{x_i}^2$ is the Laplacian operator and $\nabla = (\partial_{x_1}, \cdots, \partial_{x_{N+1}})$ is the gradient operator and Ψ_0 is defined as:

$$\Psi_0(x) = \int_{\mathbb{R}^{N(N+1)/2}} \exp\Bigg\{ - \sum_{1 \le i \le k \le N} (e^{T_{k,i} - T_{k+1,i}} + e^{T_{k+1,i+1} - T_{k,i}}) \Bigg\} dT.$$

Comparing (5) with the definition of Dyson Brownian motion, we can see that the diffusion terms are the same. To prove the main theorem II.1, we only need to prove the convergence of the drift term of (5) to that of Dyson Brownian motion. Specifically, it suffices to prove that

$$\lim_{\eta \to \infty} \eta \partial_{x_i} \log \Psi_0(\eta x) \; = \sum_{1 \le j \le N+1, j \ne i} \frac{1}{x_j - x_i}, \quad (6)$$

Note that the drift term on the left-hand side can be written as

$$\eta \partial_{x_i} \log \Psi_0(\eta x) = \frac{\eta \partial_{x_i} \Psi_0(\eta x)}{\Psi_0(\eta x)} \quad (7)$$

Next, we will analyze the denominator and numerator of (7).

A. Convergence of the denominator of (7)

By the definition of (4), we have

$$\lim_{\eta\to\infty}\exp\Big\{-\sum_{1\le j\le k\le N}(e^{\eta(T_{k,j}-T_{k+1,j})}+e^{\eta(T_{k+1,j+1}-T_{k,j})})\Big\}$$
$$=\prod_{1\le j\le k\le N}\mathbb{1}_{T_{k,j}-T_{k+1,j}<0}\cdot\mathbb{1}_{T_{k+1,j+1}-T_{k,j}<0}$$

where $\mathbb{1}_{(\cdot)}$ denotes the indicator function. Since the integrand is bounded, we have

$$\lim_{\eta\to\infty}\Psi_0(\eta x)$$
$$=\int\prod_{1\le j\le k\le N}\mathbb{1}_{T_{k,j}-T_{k+1,j}<0}\cdot\mathbb{1}_{T_{k+1,j+1}-T_{k,j}<0}\;dT$$
$$=\frac{\prod_{1\le j<j'\le N+1}(x_j-x_{j'})}{\prod_{1\le k\le N}k!}. \tag{8}$$

The above equality (8) can be proved inductively using

$$\int_{x_1>y_1>x_2>\cdots>y_k>x_{k+1}}h_k(y)dy=\frac{1}{k!}h_{k+1}(x) \tag{9}$$

with $h_k(t)=\prod_{1\le j<j'\le k}(t_j-t_{j'})$, which can be further proved by writing $h_N(y)$ as the determinant of $[y_j^{k-1}]_{1\le j\le N,1\le k\le N}$.

B. Convergence of the numerator of (7)

It can be calculated that

$$\eta\partial_{x_i}\Psi_0(\eta x)$$
$$=\int\exp\Big\{-\sum_{1\le j\le k\le N}(e^{\eta(T_{k,j}-T_{k+1,j})}+e^{\eta(T_{k+1,j+1}-T_{k,j})})\Big\}$$
$$\cdot(e^{\eta(y_i-x_i)}-e^{\eta(x_i-y_{i-1})})\eta dT$$
$$=\int\exp\Big\{-\sum_{1\le k\le N}G_k(\eta T)\Big\}e^{\eta(y_i-x_i)}\eta dT \tag{10}$$
$$-\int\exp\Big\{-\sum_{1\le k\le N}(G_k(\eta T))\Big\}e^{\eta(x_i-y_{i-1})}\eta dT. \tag{11}$$

We first derive the limit of the first part in (10). Since $K_\eta(t)=\exp\{-e^{\eta t}\}e^{\eta t}\eta$ is a delta kernel as $\eta\to\infty$, we have

$$(10)=\lim_{\eta\to\infty}\int\prod_{1\le j\le N,j\ne i}\mathbb{1}_{y_j-x_j<0}\prod_{1\le j\le N}\mathbb{1}_{x_{j+1}-y_j<0}$$
$$\cdot\exp\Big\{-\sum_{1\le k<N}G_k(\eta T)\Big\}K_\eta(y_i-x_i)\Pi dy_j$$
$$=\int\prod_{1\le j\le N,j\ne i}\mathbb{1}_{y_j-x_j<0}\prod_{1\le j\le N}\mathbb{1}_{x_{j+1}-y_j<0}$$
$$\cdot\Big(\lim_{\eta\to\infty}\int\exp\Big\{-\sum_{1\le k<N}G_k(\eta T)\Pi_{1\le k\le N-1}\Pi_{i=1}^k dT_{k,i}\Big\}\Big)$$
$$\cdot\mathbb{1}_{y_i=x_i}\prod_{j\ne k}dy_j$$
$$=\int\frac{1}{\prod_{k<N}k!}\prod_{1\le j\le N,j\ne k}\mathbb{1}_{y_j-x_j<0}\prod_{1\le j\le N}\mathbb{1}_{x_{j+1}-y_j<0}$$
$$\cdot h_N(y)\mathbb{1}_{y_i=x_i}\prod_{j\ne k}dy_j. \tag{12}$$

where the last equality is obtained by

$$\lim_{\eta\to\infty}\int\exp\{-\sum_{1\le k<N}G_k(\eta T)\}dT$$
$$=\frac{1}{\prod_{1\le k<N}k!}\prod_{1\le j<j'\le N}(T_{N,j}-T_{N,j'})$$
$$=\frac{1}{\prod_{1\le k<N}k!}\prod_{1\le j<j'\le N}(y_j-y_{j'})=\frac{1}{\prod_{1\le k<N}k!}h_N(y).$$

Similarly, the limit of (11) can be proved to be

$$-\int\frac{1}{\prod_{1\le k<N}k!}\prod_{1\le j\le N}\mathbb{1}_{y_j-x_j<0}\prod_{1\le j\le N,j\ne k-1}\mathbb{1}_{x_{j+1}-y_j<0}$$
$$\cdot h_N(y)\mathbb{1}_{y_{i-1}=x_i}\prod_{j\ne k-1}dy_j. \tag{13}$$

Combining the results in (12) and (13), which are respectively the limits of (10) and (11), we conclude that $\lim_{\eta\to\infty}\partial_{x_i}\Psi_0(\eta x)$ is equal to the partial derivative of

$$\int\frac{1}{\prod_{1\le k<N}k!}\prod_{1\le j\le N}\mathbb{1}_{y_j-x_j<0}\prod_{1\le j\le N}\mathbb{1}_{x_{j+1}-y_j<0}$$
$$\cdot h_N(y)\prod_{1\le j\le N}dy_j,$$

which is exactly (8). In other words, the limit of the numerator in (7) is equal to

$$\partial_{x_i}\frac{\prod_{1\le j<j'\le N+1}(x_j-x_{j'})}{\prod_{1\le k\le N}k!}.$$

C. Proof of the main theorem

Bringing the results in Subsections 8 and III-B into (7), we have

$$\lim_{\eta\to\infty}\frac{\partial_{x_i}\Psi_0(\eta x)}{\Psi_0(\eta x)}=\frac{\partial_{x_i}\prod_{1\le j<j'\le N+1}(x_j-x_{j'})}{\prod_{1\le j<j'\le N+1}(x_j-x_{j'})}$$
$$=\sum_{1\le j\le N+1,j\ne i}\frac{1}{x_j-x_i}.$$

Therefore, we conclude the proof of the convergence in (6) and thus the main theorem.

IV. Another Proof of Theorem II.1 Using Transition Kernels

We provide another proof of Theorem II.1. The proof will be based on a recent result in [2], which showed that we could understand the Whittaker 2d growth model at the level N as the result of applying transition kernels from previous levels. Meanwhile, Dyson Brownian motion can be seen as the result of applying the transition kernel from previous levels to level N in a multilevel Dyson Brownian motion [2], [13]. The above observations motivate the following solution. Suppose that they start from the same single Brownian motion at level 1. To prove scaled Whittaker 2d growth models converges to the Dyson Brownian motion, we only need to verify that the transition kernel of the former converges to the latter.

A. Existing results on transition kernels

We let $y = (y_i^k, 1 \le i \le k \le N) \in \mathbb{R}^{N(N+1)/2}$, $x = (x_i, 1 \le i \le N+1) \in \mathbb{R}^{N+1}$, and $r = (y, x) \in \mathbb{R}^{(N+1)(N+2)/2}$ is the concatenation of x and y. Following the notation in [2], the transition kernel of Whittaker 2d growth model is defined by

$$\Gamma(x, y) = \frac{\exp\{-T_2(r)\}}{\Psi_0(y)}, \tag{14}$$

where

$$T_2(r) = \sum_{1 \le i \le k \le N} \exp\{r_i^k - r_i^{k+1}\} + \exp\{r_{i+1}^{k+1} - r_i^k\}.$$

For the transition kernel of the Dyson Brownian motion, we first consider the Gelfand-Tsetlin cone

$$\overline{\mathcal{G}^N} = \{r = (r_i^k, 1 \le i \le k \le N) \in \mathbb{R}^{N(N+1)/2} : r_{i-1}^{k-1} \le r_i^k \le r_i^{k-1}\}.$$

Its subspace

$$D^N = \{r \in \overline{\mathcal{G}^N} : r_i^k < r_{i+1}^k, 1 \le i \le k \le N\}$$

is the domain of the process of multilevel Dyson Brownian motion. For $x \in \mathbb{R}^{N+1}$, and $x_1 < x_2 < \cdots < x_{N+1}$, we define the cross-section

$$D^N(x) = \{y \in D^N : x_1 \le y_1 \le x_2 \le y_2 \le \cdots y_N \le x_{N+1}\}.$$

The kernel for the multilevel Dyson Brownian motion is defined by

$$\Gamma_0(x, y) = \prod_{k=1}^{N} k! \prod_{1 \le i < j \le N} (x_i - x_j)^{-1} 1_{D^N(x)}(y). \tag{15}$$

Hence, we need to prove Γ converges to the transition kernel of Multilevel Dyson Brownian motion Γ_0.

B. Convergence of scaled transformation operator

Recall that for the $\Gamma(y, x)$ in (14), the numerator is $\exp\{-T_2(\eta r)\}$ and $r \in \mathbb{R}^{(N+1)(N+2)/2}$ can be written as $r = (y, x)$. It can be verified that the limit of scaled numerator satisfies

$$\lim_{\eta \to \infty} T_2(\eta r) = \begin{cases} 0 & \text{when } y \in D^N(x) \\ +\infty & \text{otherwise.} \end{cases}$$

Therefore,

$$\lim_{\eta \to \infty} \exp\{-T_2(\eta r)\} = \mathbb{1}_{y \in D^N(x)}.$$

For the denominator of (14), which is $\Psi_0(x)$, the limit has been calculated in (9) to be

$$\lim_{\eta \to \infty} \Psi_0(\eta x) = \frac{1}{\prod_{1 \le k \le N} k!} \prod_{i<j} (x_i - x_j).$$

This implies that the limit of $\Gamma(x, y)$ converges to the transformation of $\Gamma(x, y)$ of Dyson Brownian motion in (15).

V. Conclusion

In this work, we proved that scaled Whittaker 2d growth models can converge to the Dyson Brownian motion. An interesting future problem is to emulate the current proof to show that hierarchical Whittaker 2d growth models converge in distribution to the multilevel Dyson Brownian motion.

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